A "by hand" proof of the Ramanujan 6, 8, 10 power formula

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Note: Some items of literature work remain to be done. The cited Berndt and Bhargava reference, and others, need to be obtained.

1 Introduction

The number theoretic results found by Ramanujan are some of the most obscure and most interesting results in mathematics. Ramanujan would produce interesting and amazing results, often seemingly by "pulling them out of a hat".

The 6, 8, 10 "power formula" in Ramanujan's third notebook (TB) remains a mystery to mathematicians: even years after Ramanujan's life. This result has been proved using computer algebra systems, but in some circles such "computer proofs" are considered to be lacking in insight.

In this very short note, such a "by hand" proof of the Ramanujan TB formula is shewn.

2 Third Notebook Identity

The third notebook (TB)[1] result of Ramunujan states that if ${F_n}^2$ is defined as

$$F_n(a,b,c,d) = (a+b+c)^n + (b+c+d)^n - (c+d+a)^n - (d+a+b)^n + (a-d)^n - (b-c)^n$$

and if $a\,d = b\,c$ then

$$64\,F_6(a,b,c,d)\,F_{10}(a,b,c,d) = 45\,F_8^2(a,b,c,d)$$

for all $(a,b,c,d) \in C^4$ where C is the field of complex numbers.

It is possible to prove this formula using modern symbolic algebra systems. Such a proof has been performed by us. Such symbolic proofs, although able to verify a result, still leave some uncertainty in how the result itself comes about. The next section will prove this result using methods that are all reasonable to perform by hand.

[1]Berndt, Bruce C, Bhargava, S., A remarkable identity found in Ramanujan's third notebook, *Glascow Math. J.*, **34**, no. 3, 341-345.

[2]The F_n notation is used in the stated reference, and is not to be confused with the hypergeometric functions.

3 Proof

Consider the above formula in terms of primed coordinates a', b', c', and d'. A trick in the TB formula is the requirement that $a'd' - b'c' = 0$. To avoid this difficulty, it is convenient to transform from the TB or primed coordinate system into a coordinate system in which this property is automatic.

Consider the following transformations from a, b, and e into "primed coordinates"[3] given by

$$\begin{aligned} a' &= \ 3\,a + b \\ b' &= \ -e\,(3\,a + b) \\ c' &= \ -1(2\,a + 3\,b) \\ d' &= \ e\,(2\,a + 3\,b)\ . \end{aligned}$$

It follows that $a'd' - b'c' = 0$ for all a, b and e.

It is argued that except for a singular point this transformation is one-to-one. Clearly given any a, b, and e, the primed coordinates are easily obtained.

Conversely given any a', b', c', and d' such that $a'd' - b'c' = 0$, the corresponding a, b, and e are given by

$$\begin{aligned} a &= \tfrac{1}{7}\,(3\,a' + c') \\ b &= \tfrac{1}{7}\,(-2\,a' - 3\,c') \\ e &= -b'/a'\ . \end{aligned}$$

The only restriction in this transformation is that $a' \neq 0$ is assumed, or equivalently that $b \neq -3\,a$.

The transformed F_n function follows using the above formulas follows by straightforward algebra

$$\begin{aligned} F_m(a', b', c', d') \ &= (b + a\ (3 - 2\,e) - 3\,b\,e)^m \\ &+(-\,(a\ (2 + e)) + b\ (-3 + 2\,e))^m \\ &+(b\ (2 + e) + a\ (-1 + 3\,e))^m \\ &-(2\,a + 3\,b - (3\,a + b)\ e)^m \\ &-(a\ (-3 + e) - b\ (1 + 2\,e))^m \\ &-(a + 2\,a\,e + b\ (-2 + 3\,e))^m\ . \end{aligned}$$

The TB formula follows by observing that the F_n functions have the elegant factorizations given next. These results may appear difficult, but it is shewn that they are obtainable by straightforward algebra.

$$\begin{aligned} F_6(a', b', c', d') \ &= 3\ (a - 2\,b)\ (4\,a - b)\ (3\,a + b)\ (2\,a + 3\,b)\ (5\,a + 4\,b)\ (a + 5\,b) \\ &\quad (-2 + e)\ (-1 + e)\ e\ (1 + e)\ (-1 + 2\,e) \\ F_8(a', b', c', d') \ &= 56\ (a - 2\,b)\ (4\,a - b)\ (3\,a + b)\ (2\,a + 3\,b)\ (5\,a + 4\,b)\ (a + 5\,b) \\ &\quad \left(a^2 + a\,b + b^2\right)\ (-2 + e)\ (-1 + e)\ e\ (1 + e)\ (-1 + 2\,e)\ \left(1 - e + e^2\right) \\ F_{10}(a', b', c', d') \ &= 735\ (a - 2\,b)\ (4\,a - b)\ (3\,a + b)\ (2\,a + 3\,b)\ (5\,a + 4\,b)\ (a + 5\,b) \\ &\quad \left(a^2 + a\,b + b^2\right)^2\ (-2 + e)\ (-1 + e)\ e\ (1 + e)\ (-1 + 2\,e)\ \left(1 - e + e^2\right)^2\ . \end{aligned}$$

[3] This transformation is a generalization of a transformation used by Rajesh Ramamurthi in obtaining Ramamurthi fractions. In particular, Ramamurthi's transformation uses $e = 4$. (personal correspondence).

It is argued how these factorizations are seen to occur. Consider b and e fixed. Then the F_n of the last two equations are seen to be n^{th} degree polynomials in a. To shew equality of two polynomials in a, it is sufficient to shew that 1. the polynomials in a are of equal order, 2. that the zeros of the polynomials coincide, and 3. that the lagging coefficients are the same. These points are discussed in turn.

First, it is clear that the above equations are 6^{th}, 8^{th}, and 10^{th} degree polynomials in a, respectively.

Next it is seen that the zeros (values of a) are given by $a = 2\,b$, $a = b/4$, and so on. $(-1)^m = 1$ is used. The complex zeroes of $a^2 + a\,b + b^2 = 0$ are also easy. Then these specific values of a are substituted into the next to the previous F_m equation to verify that the result is zero. The double zero is verified by differentiation with respect to a and then shewing the result to be zero at the root. Each step thus follows straightforwardly, and thusly the zeroes of the equations are shewn.

The last step is to verify that the lagging coefficients of both polynomials are the same. This is done by considering polynomials in e with a and b considered fixed. The leading coefficient of e^m is verified to be zero, so the order of the polynomial in e is $m - 1$. Next the values of e resulting in zero values, (which are $e = 2$, $e = 1$, $e = 0$, $e = -1$, $e = 1/2$, $e = (1+i\sqrt{3})/2$, and $e = (1-i\sqrt{3})/2$), are substituted into the next to the last equation to verify that a zero value occurs. The multiple zero is proved for $m = 10$, as above, by differentiation with respect to e and substitution of the values $e = (1 + i\sqrt{3})/2$ and $e = (1 - i\sqrt{3})/2$. In this manner, the factorizations with respect to e are proved.

The overall constants of 3, 56, and 735 are verified by selecting specific values such as: $a = 1$, $b = 0$, $e = 4$.

With the proof of the last equation complete, the final desired result

$$64\,F_6(a', b', c', d')\,F_{10}(a', b', c', d') = 45\,F_8^2(a', b', c', d')$$

follows by inspection in the a', b', c', d' primed coordinate system.

Finally the assumption $a' \neq 0$ can be removed by taking a limit as a' approaches zero, with cognizance that $a'd' - b'c' = 0$ is always satisfied, and realizing that the F_m, being polynomials in a', are continuous functions of a'.

The desired result is shewn.

4 Discussion

A proof which can be done by hand of the famous Ramanujan 6, 8, 10 formula has been given.

A key to the proof was the factorization of the F_m formulas in the primed coordinate system. These factorizations are deep and mysterious results, perhaps as deep and mysterous as the original Ramanujan result itself. These factorizations result in an easy proof of the TB formula.

The 6, 8, 10 result is also true for $(a, b, c, d) \in F^4$, where F is any field. This more general result is not proved here.

Also see notes in the web page `http://www.georgeschils.com/ferego/docs/ramanujan/index.html` discussing and comparing this proof to one given by Berndt.